International Journal of Thermophysics, Vol. 18, No. 3, 1997

Thermomechanical Coupling in Frictionally Heated Circular Couette Flow

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Received September 23, 1996

The thermomechanical coupling in circular Couette rheometers is investigated. Asymptotic solutions of second order in the Brinkman number, Br, are developed for Newtonian fluids whose viscosity and thermal conductivity can be expressed as quadratic functions of temperature. The derived solutions are validated by comparison to previously published series solutions as the limit of planar flow is approached as well as to numerical solutions and are found to be reliable for a practical range of the Nahme number. These solutions are explicit in Br and in the properties of the fluid and thus provide valuable insight into the functionality of the relevant dependences, something that is lost in purely numerical solutions.

KEY WORDS: circular Couette flows; rheology; viscous heating.

1. INTRODUCTION

The interaction between viscous heating and flow is of importance in a number of applications involving flow of materials with temperature-dependent properties. These include polymer processing [1–3], tribology and lubrication [4], food processing [5, 6], and viscometry [7–10]. In the latter area, viscous heating is always a possible, and frequently significant, source of error in viscometric measurements at high shear rates, particularly with rotational viscometers where the entire sample is sheared continuously during the measurement. Many attempts have been made to obtain analytical solutions, including the effects of viscous heating and temperature-dependent material properties, for combined flow and heat transfer in rotational viscometers. A review of early work in the field has been

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0195-928X/97/0500-0825$12.50/0

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given in Ref. 11. When the thermal conductivity of the fluid is constant and the viscosity an exponential function of temperature, an analytical solution for planar Couette flow has been derived by Nahme [12] and quoted by Turian and Bird [9]. Gavis and Lawrence [13] revisited the Couette flow problem and, after formulating it in terms of the shear stress, determined that two solutions for velocity and temperature exist at each value of the shear stress below a critical level (which they determined for each case; see also Ref. 14), one solution at the critical stress and no solution above it. It can be shown that the lower branch of their solution coincides with the result of Nahme [12], which has the advantage of being presented in an explicit form. The fact that two solutions exist below the critical stress poses no problem from the physical viewpoint, since, as the stability analysis of Joseph [15, 16] has shown, the solutions corresponding to the upper branch (the one predicting higher temperatures) are unstable. This means that in analyzing equilibrium experimental data, only the lower (stable) branch of the solution needs to be considered. Higher temperatures (corresponding to the upper branch) may develop in practice, but, being unstable, they will tend to revert to those corresponding to the stable branch. Gavis and Lawrence [13] proceeded to present solutions for both planar and circular Couette flows of a fluid with constant thermal conductivity and viscosity described as an exponential function of temperature.

The assumption of constant thermal conductivity is common in all previous analytical results and has been justified by necessity (no closed-form solutions exist when the thermal conductivity of the fluid is temperature dependent) as well as by the relative temperature invariance of the thermal conductivity of polymer melts and commonly used fluids. However, there are a number of applications, particularly in the food industry or in applications involving phase transformations coupled with flow, in which this may not be necessarily a valid assumption. Bird et al. [7], Turian and Bird [9], and Turian [17] have presented a methodology (attributed to Broer [18]) for obtaining approximate analytical solutions, in the form of series expansions in the Brinkman number Br, to the problem of combined flow and heat transfer for materials whose viscosity and thermal conductivity are polynomial functions of temperature. They then proceeded to develop such solutions, up to second order in Br, in the case of planar Couette flow and derived formulae for the associated corrections in cone-and-plate rheometry.

The present contribution applies the same method and develops series solutions, up to second order in Br, for circular Couette flow. This device, frequently called wide-gap Couette, has received attention recently in the study of microstructure evolution during processing of concentrated suspensions [19–22] and is an excellent candidate for the study of fiber motion in nonhomogeneous flow fields [23, 24]. Many of the test fluids used in such studies are Newtonian, with a relatively high, temperature-dependent viscosity, and there is a need for explicit solutions for the velocity field which will take into account viscous heating and the effect of temperature on the transport properties of the fluid. Such solutions can be linked, as decoupled explicit modules, to numerical, microstructure-oriented models, such as the ones presented by Phillips et al. [19] for particulate suspensions or by Phan-Thien and Graham [25] and Ranganathan and Advani [26] for the evolution of the orientation in fiber suspensions. Section 2 gives an outline of the mathematical model, the method of solution, and the series solutions obtained. Section 3 is concerned with the validation of the series solutions using well-tested numerical algorithms and presents results for the distribution of velocity and temperature for various values of geometrical and flow parameters.

2. THE MATHEMATICAL MODEL

We consider steady, incompressible flow in the Couette device shown in Fig. 1. This device consists of two concentric cylinders of radii R and κR ($\kappa < 1$), the inner one of which is stationary and the outer is rotating with constant angular velocity Ω. The only nonzero velocity component in this geometry is the tangential velocity u_θ, there is no tangential pressure drop, and the equations of motion and energy reduce to

$$\frac{1}{x^2}\left\{\frac{\partial}{\partial x}\left\{\frac{\mu}{\mu_0}x^3\frac{\partial}{\partial x}\left(\frac{u}{x}\right)\right\}\right\}=0 \tag{1}$$

$$\frac{1}{x}\frac{\partial}{\partial x}\left\{\frac{k}{k_0}x\frac{\partial\Theta}{\partial x}\right\}+\mathrm{Br}\frac{\mu}{\mu_0}\left[x\frac{\partial}{\partial x}\left(\frac{u}{x}\right)\right]^2=0 \tag{2}$$

where the following nondimensionalization has been applied:

$$\Theta=\frac{T-T_0}{T_0},\qquad x=\frac{r}{R},\qquad u=\frac{u_\theta}{\Omega R},\qquad \mathrm{Br}=\frac{\mu_0(\Omega R)^2}{k_0 T_0} \tag{3}$$

In Eq. (3), Br is the Brinkman number, which is a measure of the heat generated by viscous heating as compared to the heat conducted through the material and T_0 is a reference temperature. Closed-form solutions to Eqs. (1) and (2) have been obtained for certain limiting cases ([7, 9, 13] and references therein). In this work we are interested in obtaining asymptotic analytical solutions to Eqs. (1) and (2) for fluids whose transport properties, namely, the viscosity and the thermal conductivity,

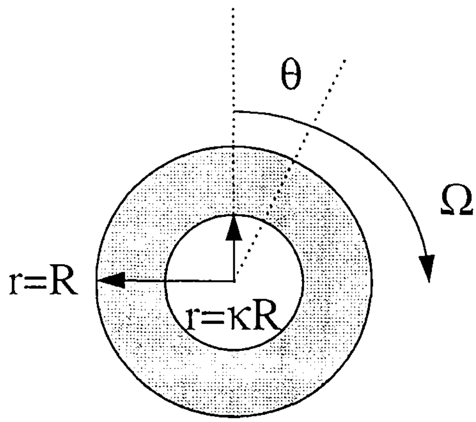

Fig. 1. Schematic description of the Couette apparatus.

are arbitrary polynomial functions of temperature. Such dependences can be expressed around a reference temperature T_0 as

$$\frac{k}{k_0} = 1 + \sum_{i=1}^{I} \alpha_i \Theta^i \tag{4}$$

$$\frac{\mu_0}{\mu} = 1 + \sum_{i=1}^{I} \beta_i \Theta^i \tag{5}$$

where the order of the approximations are not necessarily equal and where, in practice, the coefficients α_i and β_i will be determined by fitting experimental data. This presentation of material property data is very common in the process industries and has been a motivation for the development of the solutions presented in this study. The subscript (0) in Eqs. (4) and (5) indicates (known) property values corresponding to the reference temperature T_0. We consider the following boundary conditions.

At $x = \kappa$ (inner cylinder surface): $u = 0$ and $\partial\Theta/\partial x = 0$ (6a)

At $x = 1$ (outer cylinder surface): $u = 1$ and $\Theta = 0$ (6b)

It should be noted that the solution procedure presented in this work is quite general and can admit derivative as well as nonzero Dirichlet conditions on either of the cylinder surfaces. A solution to this problem with isothermal walls has been presented in Ref. 27.

2.1. Method of Solution

We seek approximate analytical solutions to Eqs. (1) and (2) subject to boundary conditions Eqs. (6a) and (6b), for fluids whose transport properties are described by Eqs. (4) and (5). Such solutions can be formulated as perturbations with respect to the Brinkman number Br [7, 9] as follows:

$$\frac{u(x)}{x}=u_0(x)+\sum_{n=1}^{N} u_n(x)\,\mathrm{Br}^n \tag{7}$$

$$\Theta(x)=\Theta_0(x)+\sum_{n=1}^{N} \Theta_n(x)\,\mathrm{Br}^n \tag{8}$$

In the absence of viscous heating (Br = 0) the system is isothermal and therefore $\Theta_0(x)$ in Eq. (8) is identically zero. The objective of the solution procedure is to determine the cofficient functions $u_n(x)$ and $\Theta_n(x)$. This is outlined in the following two sections.

2.1.1. The Momentum Equation

Substitution of the velocity profile from Eq. (7) into the momentum equation [Eq. (1)] yields

$$\frac{\partial}{\partial x}\left[x^3\frac{\mu}{\mu_0}\left(\sum_{n=0}^{N}\mathrm{Br}^n\frac{\partial u_n(x)}{\partial x}\right)\right]=0 \tag{9}$$

Integrating Eq. (9) once and expanding the integration constant (C) as a polynomial in Br yields

$$\sum_{n=0}^{N}\mathrm{Br}^n\frac{\partial u_n(x)}{\partial x}=\frac{1}{x^3}\frac{\mu_0}{\mu}\sum_{m=0}^{N}C_m\mathrm{Br}^m=\frac{1}{x^3}\left(1+\sum_{i=1}^{I}\beta_i\Theta^i\right)\sum_{m=0}^{N}C_m\mathrm{Br}^m \tag{10}$$

where the coefficients $C_0, C_1, ..., C_N$ are constants to be determined and where the dependence of viscosity on temperature was taken into account. Equating equal powers of Br on the left- and right-hand sides of Eq. (10) results in $(N+1)$ ordinary differential equations for the $(N+1)$ functions $u_n(x)$. The first three of these equations, corresponding to a solution of second order in Br, follow

$$\frac{\partial u_0}{\partial x}=\frac{C_0}{x^3} \tag{11}$$

$$\frac{\partial u_1}{\partial x}=\frac{\beta_1\Theta_1(x)\,C_0+C_1}{x^3} \tag{12}$$

$$\frac{\partial u_2}{\partial x}=\frac{[\beta_1\Theta_2(x)+\beta_2\Theta_1^2(x)]\,C_0+\Theta_1(x)\,\beta_1 C_1+C_2}{x^3} \tag{13}$$

2.1.2. *The Energy Equation*

A first integration of the momentum equation yields

$$\frac{\partial}{\partial x}\left(\frac{u}{x}\right)=\frac{1}{x^3}\frac{\mu_0}{\mu}\sum_{m=0}^{N} C_m \mathrm{Br}^m \tag{14}$$

Substituting this result into the energy equation yields

$$\frac{1}{x}\frac{\partial}{\partial x}\left\{\frac{k}{k_0}x\frac{\partial \Theta}{\partial x}\right\}+\mathrm{Br}\frac{\mu_0}{\mu}x^2\left\{\frac{1}{x^3}\sum_{m=1}^{N} C_m \mathrm{Br}^m\right\}^2=0 \tag{15}$$

Taking into account the temperature dependence of κ and μ and expressing Θ as a power series in Br [Eq. (8)] results in

$$\frac{1}{x}\frac{\partial}{\partial x}\left\{\left(1+\sum_{i=1}^{I}\alpha_i\Theta^i\right)x\sum_{i=1}^{I}\mathrm{Br}^i\frac{\partial\Theta_i(x)}{\partial x}\right\}$$
$$+\mathrm{Br}\left(1+\sum_{i=1}^{I}\beta_i\Theta^i\right)x^2\left\{\frac{1}{x^3}\sum_{m=1}^{N} C_m \mathrm{Br}^m\right\}^2=0 \tag{16}$$

As before, equating the coefficients of equal powers of Br results in N differential equations for the N functions $\Theta_n(x)$. The first two of these, corresponding to a solution of second order in Br, follow

$$\frac{\partial}{\partial x}\left[x\frac{\partial\Theta_1}{\partial x}\right]=\frac{-C_0^2}{x^3} \tag{17}$$

$$\frac{\partial}{\partial x}\left[x\frac{\partial\Theta_2}{\partial x}+\alpha_1\Theta_1(x)x\frac{\partial\Theta_1}{\partial x}\right]=\frac{-1}{x^3}[2C_0C_1+\beta_1\Theta_1(x)\,C_0^2] \tag{18}$$

2.2. The Second-Order Series Solution

Analytical integration of the $(2N+1)$ ODEs described by Eqs. (11)–(13), (17), and (18) yields approximations for the velocity and temperature profiles inside the circular Couette. Because the expressions to be integrated include powers of x, terms containing $\ln(x)$, as well as products thereof, the algebraic complexity of the solutions increases significantly with the order of the expansion in Br. The solution for the circular Couette flow with $N=2$, follows.

Velocity:

$$u_0(x) = \frac{(\kappa^2 - x^2)}{(-1+\kappa^2)\,x^2} \tag{19}$$

$$u_1(x) = \left[\frac{1}{x^2} - \frac{(-1+\kappa^4 - 4\ln(\kappa) + 4\ln(x) - 4\ln(x)\,\kappa^2)}{\kappa^2(\kappa^2-1)}\right]\frac{\beta_1 C_0^3}{16x^2} + U_1 \tag{20a}$$

with

$$U_1 = \frac{1}{16}\beta_1 C_0^3 \frac{(\kappa^2 - 1 - 4\ln(\kappa))}{\kappa^2(-1+\kappa^2)}, \qquad C_0 = \frac{2\kappa^2}{1-\kappa^2} \tag{20b}$$

$$u_2(x) = u_{2\alpha}(x) + u_{2\beta}(x) + u_{2\gamma}(x) \tag{21}$$

where

$$u_{2\alpha}(x) = \frac{1}{16} C_0^5 \frac{(\beta_1\alpha_1 - 2\beta_2)}{\kappa^4 x^2}\ln(x)^2 \tag{22}$$

$$u_{2\beta}(x) = \left[\frac{1}{16}\frac{(2x^2\kappa^2 - \kappa^2 - 2x^2)}{x^4\kappa^4}\beta_2 + u_{2\beta 1}(x)\,\beta_1^2 - \frac{1}{32}\alpha_1\frac{(2x^2\kappa^2 - \kappa^2 - 2x^2)}{x^4\kappa^4}\beta_1\right]\ln(x)\,C_0^5 \tag{23}$$

with

$$u_{2\beta 1}(x) = \frac{-1}{32}\frac{(8x^2\ln(\kappa) + 4x^2\ln(\kappa)\,\kappa^2 - 7x^2\kappa^2 + 6x^2 + x^2\kappa^4 - \kappa^2 + \kappa^4)}{\kappa^4(-1+\kappa^2)\,x^4} \tag{24}$$

$$u_{2\gamma}(x) = \frac{-1}{384x^6}(u_{2\gamma 1}x^6 + u_{2\gamma 2}x^4 + u_{2\gamma 3}x^2 + u_{2\gamma 4}) \tag{25}$$

The coefficients $u_{2\gamma 1}$–$u_{2\gamma 4}$ in Eq. (25) are given in Appendix A as functions of (κ) and of the fluid properties. The zeroth-order result, u_0, coincides, as expected, with the well-known solution for circular Couette flow of a fluid with temperature-independent properties.

Temperature:

$$\Theta_1(x) = \left[\frac{-1}{4}\frac{(2\ln(x) - \kappa^2)}{\kappa^2} - \frac{1}{(4\cdot x^2)}\right]C_0^2 \tag{26}$$

$$\Theta_2(x) = F_1\ln(x)^2 + F_2(x)\ln(x) + F_3(x) \tag{27}$$

where the coefficients F_1–F_3 are given by

$$F_1 = \frac{-1}{8}\frac{\alpha_1}{\kappa^4} C_0^4 \tag{27a}$$

$$F_2(x) = \frac{C_0^4}{16} \frac{\left(\begin{array}{l} [[(4\ln(\kappa) + 4\ln(\kappa)\,\kappa^2 - 3\kappa^2 + 3)\,\beta_1 - 2\alpha_1\kappa^2(1-\kappa^2)]\,x^2 \\ \quad + 2\kappa^2(\kappa^2-1)(\beta_1 - \alpha_1)] \end{array} \right)}{\kappa^4 x^2(-1+\kappa^2)} \tag{27b}$$

$$F_3(x) = C_0^4 \left(\frac{x^2-1}{\kappa^2-1}\right) \frac{\left(\begin{array}{l} (4x^2 - 5x^2\kappa^2 + x^2\kappa^4 + 16\cdot x^2\ln(\kappa) - \kappa^2 + \kappa^4)\,\beta_1 \\ \quad + 2\alpha_1\kappa^2\cdot(-1+\kappa^2)(x^2-1) \end{array} \right)}{(-64)\,\kappa^2 x^4} \tag{27c}$$

The first-order result $\Theta_1(x)$ corresponds to the circular Couette flow of a fluid with temperature-independent transport properties. Evidently, the second-order result for temperature considers only linear dependence of μ on Θ [only the coefficient β_1 appears in Eqs. (26) and (27)]. For this reason, the series solution for temperature given by Eqs. (26) and (27) is valid for a narrower range of Brβ than the velocity solution. We do not consider this a critical disadvantage, since the main use of the presented solutions is anticipated to be in the interpretation of rheological data and the description of the flow field in the Couette. When a higher accuracy in temperature is needed, a third-order solution, of the form

$$\Theta(x) = \Theta_1(x)\,\mathrm{Br} + \Theta_2(x)\,\mathrm{Br}^2 + \Theta_3(x)\,\mathrm{Br}^3 \tag{28}$$

has also been derived in the context of this study. This solution offers an extended range of accuracy as compared to the second-order solution, at the cost of being significantly more complicated. The obtained expression for $\Theta_3(x)$ is shown in Appendix B.

3. VALIDATION OF THE SERIES SOLUTION

To determine the range of validity of the proposed series solution, the boundary-value problem defined by Eqs. (1) and (2) is solved numerically, subject to the boundary conditions of Eqs. (6a) and (6b) and with material properties corresponding to Eqs. (4) and (5). The governing Eqs. (1) and (2) are rendered explicit in the derivatives of u and Θ through application of the chain rule. After some manipulation, the momentum equation takes the form

$$A(x)\frac{d^2u(x)}{dx^2}+B(x)\frac{du(x)}{dx}+C(x)\,u(x)=0 \tag{29}$$

with

$$\begin{aligned}
A(x)&=x^2[1+\beta_1\Theta(x)+\beta_2\Theta^2(x)]\\
B(x)&=x\left[1+\beta_2\Theta^2(x)+\beta_1\Theta(x)-\beta_1 x\frac{d\Theta(x)}{dx}-2x\beta_2\Theta(x)\frac{d\Theta(x)}{dx}\right]\\
C(x)&=x\left[\beta_1\frac{d\Theta(x)}{dx}-\beta_1\frac{\Theta(x)}{x}-\beta_2\frac{\Theta^2(x)}{x}-\frac{1}{x}+2\beta_2\Theta(x)\frac{d\Theta(x)}{dx}\right]
\end{aligned} \tag{30}$$

while the energy equation can be written as

$$\begin{aligned}
&[\alpha_1+2\alpha_2\Theta(x)]\left(\frac{d\Theta(x)}{dx}\right)^2+[1+\alpha_1\Theta(x)+\alpha_2\Theta(x)^2]\left[\frac{1}{x}\frac{d\Theta(x)}{dx}+\frac{d^2\Theta(x)}{dx^2}\right]\\
&\qquad+\mathrm{Br}[1+\beta_1\Theta(x)+\beta_2\Theta(x)^2]^{-1}x^2\frac{d(u(x)/x)}{dx}=0
\end{aligned} \tag{31}$$

Equations (29)–(31) are discretized using central finite differences and are solved through a successive relaxation algorithm [28]. The accuracy of the numerical algorithm has been verified through detailed comparison with the analytical solution of Gavis and Lawrence [13], obtained for fluids of constant thermal conductivity.

It should be pointed at this stage that the Brinkman number alone is not sufficient to quantify the effect of viscous heating on the velocity profile in fluids with temperature-dependent viscosity. This is evident, since the velocity profile in a fluid with constant viscosity ($\beta_i=0$) will remain unaffected by viscous heating irrespective of the magnitude of the Brinkman number. The appropriate scale in this case is the Nahme number, defined as $\mathrm{Na}=\mathrm{Br}(\partial\mu/\partial T)(T_0/\mu_0)$ [29]. For this reason, viscous heating is quantified by the use of the product $\mathrm{Br}\beta$ instead of Br alone in the following discussion. Table I summarizes the results of a comparison between numerical (based on 400-node spatial discretization) and series solutions, for values of the product $\mathrm{Br}\beta$ ranging from 0.5 to 2.0 and for two values of the eccentricity parameter κ, namely, $\kappa=0.75$ and $\kappa=0.85$. Listed is the magnitude of the norm of the percentage relative differences of

Table I. The Norm of the Difference Between Numerical and Series Solutions for Flow in a Circular Couette for a Range of Values of the Product Br/β: $\beta=\beta_1=1$, $\beta_2=0.5$, $\alpha_1=0.1$, $\alpha_2=0.1$

	$\kappa=0.75$		$\kappa=0.85$	
Br/β	Norm(u_R)	Norm(Θ_R)	Norm(u_R)	Norm(Θ_R)
0.5	0.044	0.29	0.053	0.34
1.0	0.33	2.56	0.40	2.96
1.5	1.03	8.61	1.26	10.0
1.75	1.56	13.55	1.92	15.8
2.0	2.24	20.01	2.74	23.3

velocity and temperature [norm(u_R) and norm(Θ_R), respectively], which are defined as

$$\text{norm}(F_R)=\frac{100}{K}\sum_{j=1}^{K}\frac{|F_s-F_n|}{F_n} \tag{32}$$

where F is either temperature or velocity, the subscripts s and n indicate series and numerical solutions respectively, the subscript j indicates the jth point across the gap, and K is the total number of points across the gap. Evidently, the accuracy of the series solution for the velocity profile is satisfactory, with the norm of the errors being less than 1% for Br/β up to around 1.5 (depending on κ). The same is not true for the temperature profile; the norm of the errors is less than 5% only for Br/β less than 1.5. Furthermore, the maximum deviation between series and numerical solutions occurs at the location of the maximum temperature and is higher than the average errors shown in Table I; for $Br/\beta=1.0$ the maximum relative error for the temperature is 3.54%, while for $Br/\beta=1.5$ it is 11.6% ($\kappa=0.75$). A more detailed comparison between the numerical and the series solutions for velocity and temperature across the gap of the Couette for three values of κ is given in Figs. 2–4. In agreement with the trends shown in Table I, the discrepancy between numerical and analytical solutions increases slightly as the gap of the Couette becomes narrower. It can also be seen in Figs. 2–4 that the series and numerical solutions for velocity are practically indistinguishable.

Further to the validation of the presented series solution, it can be shown that formal limits (as κ approaches unity) of the expressions obtained for u_0–u_2 and Θ_1–Θ_2 [Eqs. (19)–(27)] yield the velocity and temperature profiles corresponding to planar Couette flow (obtained by Turian and Bird [9] and verified by the present author and Louwagie [30]). The

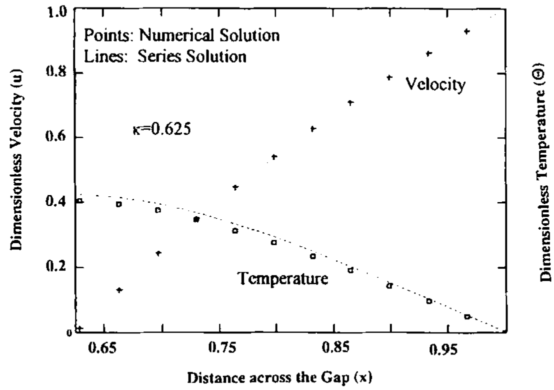

Fig. 2. Comparison between series solutions (of the second order for u and third order for Θ) and numerical solution for frictionally heated circular Couette flow. Material parameters are as in Table I, $\mathrm{Br}/\beta = 1.5$, and $\kappa = 0.625$.

derivation of these limits for u_0–u_1 and Θ_1–Θ_2 is straightforward and is omitted here for the sake of brevity. The derivation of the limit corresponding to the term u_2 is more tedious because of its algebraic complexity and proceeds as follows: The term corresponding to u_2 in planar Couette flow, $u_{2p}(\xi)$, can be written as

$$u_{2p}(\xi) = \beta_1^2 f_1(\xi) + \beta_2 f_2(\xi) + \beta_1 \alpha_1 f_3(\xi) \tag{33}$$

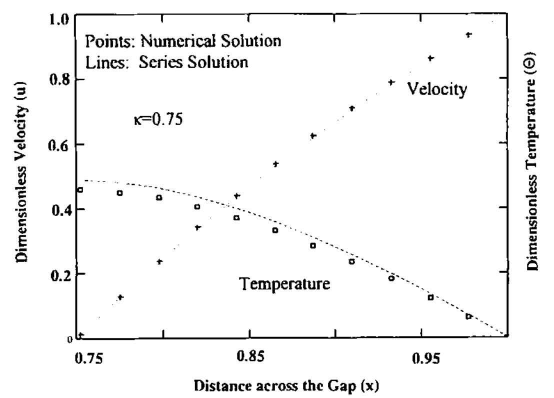

Fig. 3. Comparison between series solutions (of the second order for u and third order for Θ) and numerical solution for frictionally heated circular Couette flow. Material parameters are as in Table I, $\mathrm{Br}/\beta = 1.5$, and $\kappa = 0.75$.

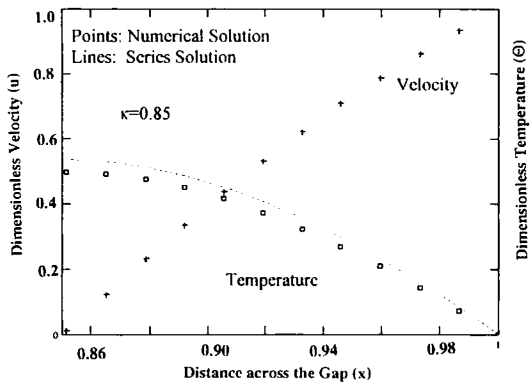

Fig. 4. Comparison between series solutions (of the second order for u and third order for Θ) and numerical solution for frictionally heated circular Couette flow. Material parameters are as in Table I, $\mathrm{Br}/\beta = 1.5$, and $\kappa = 0.85$.

where $f_1(\xi)$–$f_3(\xi)$ are polynomials in (ξ) [9, 28] and ξ is the corresponding dimensionless coordinate $\xi = (x + \kappa)/(1 - \kappa)$. Equation (21) can be recast in the form of Eq. (33) as

$$u_2(\xi) = A_1(\kappa, \xi)\,\beta_1^2 + A_2(\kappa, \xi)\,\beta_2 + A_3(\kappa, \xi)\,\beta_1\alpha_1 \qquad (34)$$

It can then be shown that

$$\lim_{\kappa \to 1}[A_1(\kappa, \xi)] = f_1(\xi), \quad \lim_{\kappa \to 1}[A_2(\kappa, \xi)] = f_2(\xi), \quad \lim_{\kappa \to 1}[A_3(\kappa, \xi)] = f_3(\xi)$$

Similar tests have been carried out for the third-order term $[\Theta_3(x)]$ shown in Appendix B. In light of the algebraic complexity of the presented series solutions, it is desirable to have an estimate of the improvement in accuracy offered by higher-order approximations. Table II shows the variation of the

Table II. Relative Percentage Errors Associated with the First-, Second-, and Third-Order Series Solutions for Velocity and Temperature for a Range of Br/β and for $\kappa = 0.625$[a]

Br/β	Norm(u_0)	Norm(u_1)	Norm(u_2)	Norm(Θ_1)	Norm(Θ_2)	Norm(Θ_3)
0.5	2.46	0.30	0.03	11.2	1.76	0.23
1.0	4.32	1.05	0.23	21.6	6.75	1.97
1.5	5.79	2.11	0.72	31.3	14.6	6.59
1.75	6.41	2.72	1.09	36.0	19.5	10.37
2.0	6.98	3.39	1.57	40.5	25.0	15.3
2.5	7.96	4.81	2.82	49.2	37.7	29.2

[a] The material parameters are as in Table I.

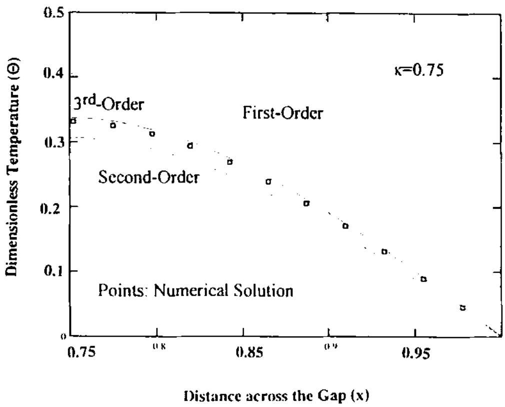

Fig. 5. Comparison between series solutions (of the first, second, and third order in Br) and the numerical solution for frictionally heated circular Couette flow. Material parameters are as in Table I. $\mathrm{Br}/\beta = 1.0$.

relative percentage error [RPE; defined as in Eq. (32)] of the zeroth-, first-, and second-order solutions for velocity and the corresponding solutions for temperature for $0.5 < \mathrm{Br}/\beta < 2.5$ and $\kappa = 0.625$. Columns 2–4, indicated as norm(u_0)-norm(u_2) are the RPEs associated with the zeroth-, first-, and second-order series solutions for velocity, while columns 5–7 are the corresponding errors of the temperature solutions. Evidently, the second-order solution for the velocity [Eqs. (19)–(25)] is accurate to within 1% RPE for $\mathrm{Br}/\beta < 1.75$, while the first-order result is accurate to within 1% RPE only for $\mathrm{Br}/\beta < 1.0$. Similar conclusions can be drawn for the temperature solutions. Figure 5 shows a further comparison of the performance of various-order series solutions for temperature, for $\mathrm{Br}/\beta = 1.0$ and $\kappa = 0.75$. The improvement offered by the third-order series solution for temperature is evident.

4. CONCLUSION

Series solutions, up to second order in the Brinkman number, Br, have been developed for circular Couette flow of materials whose viscosity and thermal conductivity can be expressed as polynomial functions of temperature with arbitrary coefficients. At this stage, no experimental data are available for comparison. However, the derived solutions have been validated by extensive comparison with numerical solutions and are found to be reasonably accurate for a practical range of the Nahme number. The norm of the relative errors in the predicted velocity profile is less than 1%

for $\mathrm{Br}\beta < 1.5$ in a Couette with $\kappa = 0.75$ and less than 1% for $\mathrm{Br}\beta < 1.2$ in a Couette with $\kappa = 0.85$. The presented solutions exhibit the anticipated asymptotic behavior as κ approaches unity and provide valuable insight into the functionality of the relevant dependences, something that is lost in purely numerical solutions.

APPENDIXES

Appendix A

The coefficients appearing in the expression for the second-order term in the velocity profile [Eqs. (21)–(25)] are as follows:

$$u_{2y1} = -384(A_2(\kappa)\,\beta_2 + A_1(\kappa)\,\beta_1^2 + A_3(\kappa)\,\beta_1 \cdot \alpha_1) \tag{A1}$$

$$u_{2y2} = C_0^5 u_{2y21} + 192 u_{2y22} \tag{A2}$$

where

$$u_{2y21} = \frac{3}{\kappa^4}\,[(-3\kappa^4 + 12\kappa^2 - 16\ln(\kappa) - 12)\,\beta_1^2 + (-2\kappa^4 - 4 + 4\kappa^2)(\beta_1\alpha_1 - 2\beta_2)] \tag{A3}$$

$$u_{2y22} = A_4(\kappa)\,\beta_1^2 + A_5(\kappa)\,\beta_1\alpha_1 - \beta_2 A_6(\kappa) \tag{A4}$$

$$u_{2y3} = \left[3\,\frac{(-5\kappa^2 + \kappa^4 + 4 + 12\ln(\kappa))}{\kappa^2(-1+\kappa^2)}\,\beta_1^2 + 3\,\frac{(-1+2\kappa^2)}{\kappa^2}\,\beta_1\alpha_1 + 6\,\frac{\beta_2}{\kappa^2} - 12\beta_2\right] C_0^5 \tag{A5}$$

$$u_{2y4} = (\beta_1^2 + 4\beta_2 - 2\beta_1\alpha_1)\,C_0^5 \tag{A6}$$

The constants $A_1(\kappa)$–$A_6(\kappa)$ are given by

$$A_1(\kappa) = \frac{1}{12}\,\kappa^6\,\frac{\begin{pmatrix} -19\kappa^4 - 12\kappa^4\ln(\kappa) + 4\kappa^6 + 26\kappa^2 + 108\ln(\kappa)\,\kappa^2 \\ -48\ln(\kappa)^2\kappa^2 - 96\ln(\kappa) - 96\ln(\kappa)^2 - 11 \end{pmatrix}}{(-1+\kappa^2)^7} \tag{A7}$$

$$A_2(\kappa) = \frac{-1}{6}\,\kappa^6\,\frac{(4\kappa^4 + 5 + 24\ln(\kappa)^2 + 36\ln(\kappa) - 24\ln(\kappa)\,\kappa^2 - 9\kappa^2)}{(-1+\kappa^2)^6} \tag{A8}$$

$$A_3(\kappa)=\frac{1}{12}\kappa^6\frac{(5-9\kappa^2+4\kappa^4+36\ln(\kappa)-24\ln(\kappa)\kappa^2+24\ln(\kappa)^2)}{(-1+\kappa^2)^6} \tag{A9}$$

$$A_4(\kappa)=\frac{-\kappa^6}{6}\frac{\begin{pmatrix}[5\kappa^8-38\kappa^6+(24\ln(\kappa)+108)\kappa^4+(-168\ln(\kappa)\\ -122+48\ln(\kappa)^2)\kappa^2+144\ln(\kappa)+96\ln(\kappa)^2+47]\end{pmatrix}}{(-1+\kappa^2)^7} \tag{A10}$$

$$A_5(\kappa)=\frac{1}{6}\kappa^6\frac{(15\kappa^4-2\kappa^6+36\ln(\kappa)-30\kappa^2-24\ln(\kappa)\kappa^2+24\ln(\kappa)^2+17)}{(-1+\kappa^2)^6} \tag{A11}$$

$$A_6(\kappa)=\frac{-1}{3}(2\kappa^6-15\kappa^4-17-24\ln(\kappa)^2$$

$$-36\ln(\kappa)+24\ln(\kappa)\kappa^2+30\kappa^2)\frac{\kappa^6}{(-1+\kappa^2)^6} \tag{A12}$$

Appendix B

The third-order term $\Theta_3(x)$ in Eq. (28) can be expressed as follows:

$$\Theta_3(x)=L_3\ln(x)^3+L_2(x)\ln(x)^2+L_1(x)\ln(x)+L_0(x) \tag{B1}$$

where the coefficients L_0–L_3 are given below:

$$L_0(x)=L_{0\alpha}\left(\frac{1}{x^2}\right)+L_{0\beta}\left(\frac{1}{x^4}\right)+L_{0\gamma}\left(\frac{1}{x^6}\right)+L_{0\delta} \tag{B2}$$

$$L_{0\alpha}=\left[\frac{1}{256}\frac{L_{0\alpha 1}}{\kappa^4(-1+\kappa^2)}-\frac{\beta_1^2}{256}\frac{(3+4\ln(\kappa)-4\kappa^2+\kappa^4)^2}{\kappa^4(-1+\kappa^2)^2}-\frac{1}{2}C_2\right]C_0^6 \tag{B2a}$$

$$L_{0\beta}=\frac{1}{256}\frac{L_{0\beta 1}}{\kappa^2(-1+\kappa^2)}C_0^6 \tag{B2b}$$

$$L_{0\gamma}=\frac{-C_0^6}{2304}(18\alpha_1^2-11\beta_1\alpha_1+4\beta_2+\beta_1^2-12\alpha_2) \tag{B2c}$$

$$L_{0\delta}=\left(J_1+\frac{1}{2}C_2\right)C_0^6 \tag{B2d}$$

$$L_1(x) = \left[L_{1\beta} + \frac{L_{1\alpha}}{x^2} + \frac{1}{128} \frac{(-6\alpha_1^2 + 6\beta_1\alpha_1 - 2\beta_2 + 4\alpha_2 - \beta_1^2)}{\kappa^2 x^4} \right] C_0^6 \tag{B3}$$

$$L_{1\alpha} = \frac{-1}{64} \frac{L_{1\alpha 1}}{\kappa^4(\kappa^2 - 1)} \tag{B3a}$$

$$\begin{aligned} L_{1\alpha 1} &= 2(\kappa^2 - 1)(2\kappa^2\alpha_2 - 2\beta_2\kappa^2 - 3\kappa^2\alpha_1^2 + 4\beta_2) \\ &\quad + (12 \ln(\kappa) - 11\kappa^2 + 9 + 4 \ln(\kappa)\, \kappa^2 + 2\kappa^4)\, \beta_1^2 \\ &\quad + (-4 \ln(\kappa)\, \kappa^2 - 3\kappa^2 - 12 \ln(\kappa) - 1 + 4\kappa^4)\, \beta_1\alpha_1 \end{aligned} \tag{B3b}$$

$$L_{1\beta} = \frac{1}{128} \frac{L_{1\beta 1}}{(-1 + \kappa^2)\, \kappa^4} + H \tag{B3c}$$

$$\begin{aligned} L_{1\beta 1} &= 2\kappa^2(\kappa^2 - 1)(-3\alpha_1^2 + 2\alpha_2) \\ &\quad + (-\kappa^4 - 24 \ln(\kappa) - 10 + 11\kappa^2 - 8 \ln(\kappa)\, \kappa^2)\, \beta_1\alpha_1 \end{aligned} \tag{B3d}$$

$$L_2(x) = \left[L_{2\alpha} + \frac{1}{32} \frac{(-2\beta_2 + 3\alpha_1\beta_1 + 2\alpha_2 - 3\alpha_1^2)}{\kappa^4 x^2} \right] C_0^6 \tag{B4}$$

$$L_{2\alpha} = \frac{(-3\kappa^2 + 3 + 4 \ln(\kappa) + 4 \ln(\kappa)\, \kappa^2)\, \beta_1\alpha_1 - \kappa^2(\kappa^2 - 1)(-3\alpha_1^2 + 2\alpha_2)}{32\kappa^6(\kappa^2 - 1)} \tag{B4a}$$

$$L_3 = \left(\frac{C_0}{\kappa} \right)^6 \frac{(2\alpha_2 - 3\alpha_1^2)}{48} \tag{B5}$$

The constants C_0, J_1, C_2, H, $L_{0\beta 1}$, and $L_{0\alpha 1}$ appearing in the expressions above are

$$C_0 = \frac{2\kappa^2}{1 - \kappa^2} \tag{B6}$$

$$J_1 = \frac{1}{2304} \frac{J_{1\alpha}}{\kappa^4(-1 + \kappa^2)^2} \tag{B7}$$

$$\begin{aligned} J_{1\alpha} &= 18\kappa^4(\kappa^2 - 1)^2\alpha_1^2 - (\kappa^2 - 1)(2\kappa^6 - 29\kappa^4 - 144\kappa^2 \ln(\kappa) \\ &\quad + 135\kappa^2 - 108)\, \beta_1\alpha_1 + (216\kappa^2 \ln(\kappa) + 621\kappa^2 - 216 \ln(\kappa) \\ &\quad - 539\kappa^4 - 26\kappa^8 + 144 \ln(\kappa)^2 + 187\kappa^6 - 243)\, \beta_1^2 \\ &\quad - 12\kappa^4(\kappa^2 - 1)^2\alpha_2 + (22\kappa^4 - 126\kappa^2 + 216)(\kappa^2 - 1)^2\, \beta_2 \end{aligned} \tag{B8}$$

$$C_2 = (\beta_1 \alpha_1 - 2\beta_2)\, C_{2\beta} + C_{2\alpha}\beta_1^2 \tag{B9}$$

$$C_{2\alpha} = \frac{\begin{pmatrix} 48(\kappa^2+2)\ln(\kappa)^2 + 24(\kappa^2-1)(\kappa^2-6)\ln(\kappa) \\ +5\kappa^8 - 38\kappa^6 + 108\kappa^4 - 122\kappa^2 + 47 \end{pmatrix}}{(-1+\kappa^2)^2\, 192\kappa^4} \tag{B9a}$$

$$C_{2\beta} = \frac{-15\kappa^4 + 2\kappa^6 - 36\ln(\kappa) + 30\kappa^2 + 24\ln(\kappa)\,\kappa^2 - 24\ln(\kappa)^2 - 17}{192\kappa^4(\kappa^2-1)} \tag{B9b}$$

$$H = \frac{1}{6}(-2\beta_2 + \beta_1\alpha_1)\, H_2 + \frac{1}{6} H_1 \beta_1^2 \tag{B10}$$

$$H_1 = \frac{1}{64}\, \frac{\begin{pmatrix} [(-48\kappa^4 - 192\kappa^2 - 96)\ln(\kappa)^2 - 24(\kappa^4-1)(\kappa^2-6)\ln(\kappa) \\ -47 + 4\kappa^6 - 61\kappa^4 + 2\kappa^8 + 102\kappa^2] \end{pmatrix}}{(-1+\kappa^2)^2\,\kappa^6} \tag{B10a}$$

$$H_2 = \frac{1}{64}\, \frac{\begin{pmatrix} [24\ln(\kappa)^2 + 24\ln(\kappa)^2\,\kappa^2 + 12\ln(\kappa)\,\kappa^2 - 24\ln(\kappa)\,\kappa^4 \\ +36\ln(\kappa) + 6\kappa^4 + 2\kappa^6 + 17 - 25\kappa^2) \end{pmatrix}}{(-1+\kappa^2)\,\kappa^6} \tag{B10b}$$

$$\begin{aligned} L_{0\alpha 1} = {} & -6\kappa^4(\kappa^2-1)\,\alpha_1^2 + (-\kappa^4 - 12 - 32\kappa^2\ln(\kappa) + \kappa^6 + 12\kappa^2)\,\beta_1\alpha_1 \\ & + 4\kappa^4(\kappa^2-1)\,\alpha_2 + (-40\kappa^2 + 20\kappa^4 + 24 - 4\kappa^6)\,\beta_2 \\ & + [5\kappa^6 - 29\kappa^4 + (60 + 16\ln(\kappa))\,\kappa^2 - 48\ln(\kappa) - 36]\,\beta_1^2 \end{aligned} \tag{B11}$$

$$\begin{aligned} L_{0\beta 1} = {} & 2(\kappa^2-1)(\beta_2\kappa^2 + 3\kappa^2\alpha_1^2 - 2\kappa^2\alpha_2 - \beta_2) \\ & + (16\ln(\kappa) - \kappa^2 + 3 - 2\kappa^4)\,\beta_1\alpha_1 \\ & + (-\kappa^4 - 3 - 8\ln(\kappa) + 4\kappa^2)\,\beta_1^2 \end{aligned} \tag{B12}$$

NOMENCLATURE

Br	Brinkman number [Eq. (3)]
k	Thermal conductivity
R	Radius
r	Radial distance
T	Temperature
T_0	Reference temperature
u	Dimensionless tangential velocity
u_0	Tangential velocity

u_0–u_2 Terms of series solution for velocity
x Dimensionless radial distance

Greek Symbols

α_i Coefficients in the thermal conductivity model [Eq. (4)]
β_i Coefficients in the viscosity model [Eq. (5)]
μ Viscosity
κ Eccentricity of the Couette (inner radius/outer radius)
Θ Dimensionless temperature
Θ_1–Θ_3 Terms of series solution for temperature
Ω Angular velocity

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